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# Logarithms of pseudodifferential operators

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In this note, we clarify the relation between operators and their exponentials. As an application, a sufficient condition for invertibility of pseudodifferential operators of infinite order is obtained.

A pseudodifferential operator is defined by a symbol  $P(x, \xi)$ . The operator defined by  $P(x, \xi)$  is denoted by  $:P(x, \xi):$ . The composite operator of two operators  $:P(x, \xi):$  and  $:Q(x, \xi):$  is written in the following form:

$$:P(x, \xi)::Q(x, \xi): = :\exp(\partial_{\xi} \cdot \partial_y) P(x, \xi) Q(y, \eta) \Big|_{\eta=\xi}^{y=x} :.$$

## 1. Composition formula.

Let us consider a symbol of the form  $\exp\{p(x, \xi)\}$ . In analytic category, such a symbol makes sense if  $p(x, \xi)$  is of order at most  $l-0$ , i.e., if  $\lim_{|\xi| \rightarrow \infty} p(x, \xi)/|\xi| = 0$ . More generally, a formal sum of symbols  $\exp\{\sum_{j \geq 0} p_j(x, \xi)\}$  makes sense if

$\exists d > 0, 1 > A > 0; \forall h > 0, \exists H > 0$  such that

$$|p_j(x, \xi)| \leq A^j (h|\xi| + H), j \geq 0, |\xi| \geq (j+1)d,$$

(then  $\sum_j p_j(x, \xi)$  is said to be a formal symbol of order  $l-0$ ).

Let us recall the composition formula for operators with exponential symbols:

Theorem 1. Let  $p(x, \xi)$  and  $q(x, \xi)$  be symbols of order 1-0.

Let  $\{w_j\}$  and  $\{r_j\}$  be sequences of symbols defined by

$$(1.1) \quad w_0(x, y, \xi, \eta) = p(x, \xi) + q(y, \eta),$$

$$(1.2) \quad w_{j+1} = \frac{1}{j+1} (\partial_\xi \cdot \partial_y w_j + \sum_{k=0}^j \partial_\xi w_k \cdot \partial_y w_{j-k}), \quad j \geq 0,$$

$$(1.3) \quad r_j(x, \xi) = w_j(x, x, \xi, \xi), \quad j \geq 0.$$

Then  $\sum_j r_j(x, \xi)$  is a formal symbol of order 1-0 satisfying

$$(1.4) \quad : \exp\{p(x, \xi)\} : : \exp\{q(x, \xi)\} : = : \exp\left\{\sum_j r_j(x, \xi)\right\} :.$$

In the preceding theorem, we can replace  $p$  and  $q$  by formal symbols  $\sum_j p_j$  and  $\sum_j q_j$  of order 1-0, respectively.

## 2. Exponential of operators and operators with exponential symbols.

Let  $\sum_j p_j(x, \xi)$  and  $\sum_j q_j(x, \xi)$  be formal symbols of order 1-0. We give an answer to the following problem: Under what conditions does the equality

$$\exp: \sum_j p_j(x, \xi) : = : \exp\left\{\sum_j q_j(x, \xi)\right\} :$$

hold ?

First part of the answer is

Theorem 2. Let  $\sum_j p_j(x, \xi)$  be a formal symbol of order 1-0. Let  $\{\psi_{l,k}^{(j)}(x, y, \xi, \eta)\}$  and  $\{q_k^{(j)}(x, \xi)\}$  be sequences

of symbols defined by

$$(2.1) \quad \psi_{l,0}^{(0)}(x,y,\xi,\eta) = p_l(x,\xi), \quad l=0,1,2,\dots,$$

$$(2.2) \quad \psi_{l,0}^{(j)}(x,y,\xi,\eta) = 0, \quad j=1,2,\dots, l=0,1,2,\dots,$$

$$(2.3) \quad q_k^{(j+1)}(x,\xi) = \frac{1}{j+1} \sum_{l=0}^k \psi_{l,k-l}^{(j)}(x,x,\xi,\xi),$$

$$(2.4) \quad \psi_{l,k+1}^{(j)}(x,y,\xi,\eta) = \frac{1}{k+1} \left\{ \partial_\xi \cdot \partial_y \psi_{l,k}^{(j)}(x,y,\xi,\eta) + \sum_{\nu=0}^l \sum_{\mu=0}^{j-1} \partial_\xi \psi_{\nu,k}^{(\mu)}(x,y,\xi,\eta) \cdot \partial_y q_{l-\mu}^{(j-\mu)}(y,\eta) \right\}.$$

Set  $q_k(s,x,\xi) = \sum_{j=1}^{k+1} s^j q_k^{(j)}(x,\xi)$  ( $s \in \mathbb{C}$ ). Then, for each  $s$ , the formal series  $\sum q_k(s,x,\xi)$  is a formal symbol of order 1-0 such that

$$(2.5) \quad \exp\left\{s: \sum_j p_j(x,\xi): \right\} = : \exp\left\{ \sum_k q_k(s,x,\xi) \right\}:$$

holds.

Example 3.  $\exp(sx \sqrt{D}) = : \exp(sx \sqrt{\xi} + \frac{s^2 x}{4}) : (n=1; D = : \xi : = \partial/\partial x, x=x_1).$

Conversely, we have

Theorem 4. Let  $\sum_j q_j(x,\xi)$  be a formal symbol of order 1-0. Let  $\{\psi_{l,k}^{(j)}(x,y,\xi,\eta)\}$  be a sequence of symbols defined by

$$(2.6) \quad \psi_{0,0}^{(0)}(x,y,\xi,\eta) = q_0(x,\xi),$$

$$(2.7) \quad \psi_{l,0}^{(j)}(x,y,\xi,\eta) = 0, \quad j=1,2,\dots, l=0,1,2,\dots,$$

$$(2.8) \quad \psi_{\ell, k+1}^{(j)}(x, y, \xi, \eta) = \frac{1}{k+1} \{ \partial_{\xi} \cdot \partial_y \psi_{\ell, k}^{(j)}(x, y, \xi, \eta) \\ + \sum_{\nu=0}^{\ell} \sum_{\mu=0}^{j-1} \sum_{i=0}^{\ell-\nu} \frac{1}{j-\mu} \partial_{\xi} \psi_{\nu, k}^{(\mu)}(x, y, \xi, \eta) \cdot \partial_y \psi_{i, \ell-\nu-i}^{(j-\mu-1)}(y, y, \eta, \eta) \},$$

$$(2.9) \quad \psi_{k, 0}^{(0)}(x, y, \xi, \eta) = q_k(x, \xi) - \sum_{j=0}^k \sum_{\ell=0}^{k-1} \frac{1}{j+1} \psi_{\ell, k-\ell}^{(j)}(x, x, \xi, \xi).$$

Set  $p_k(x, \xi) = \psi_{k, 0}^{(0)}(x, x, \xi, \xi)$ . Then  $\sum p_k(x, \xi)$  is a formal symbol of order  $l=0$  such that

$$(2.10) \quad : \exp \{ \sum_j q_j(x, \xi) \} : = \exp : \sum_k p_k(x, \xi) :$$

holds.

### 3. Invertibility for pseudodifferential operators of infinite order.

It is well known that a pseudodifferential operator of finite order is invertible if its symbol is invertible as a symbol. The same is true for infinite order case.

Theorem 5. Let  $P(x, \xi)$  be a symbol. Suppose that  $1/P(x, \xi)$  is also a symbol, i.e., for each  $h > 0$ , there is a constant  $C_h > 0$  such that

$$C_h^{-1} \exp(-h|\xi|) \leq |P(x, \xi)| \leq C_h \exp(h|\xi|).$$

Then  $:P(x, \xi):$  is invertible.